Exercise

① Qn. → If $x^3+y^3-x^3y^2z=0$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $x=y=1$.

Ans. → We have, $x^3y^2z = x^3+y^3$

$$\therefore z = \frac{x^3+y^3}{x^3y^2} = \frac{\cancel{x^3}}{\cancel{x^3}y^2} + \frac{\cancel{y^3}\,y}{x^3\cancel{y^2}}$$

Diff. partially w.r.t. x, we keeping y constant,

$$z = \frac{1}{y^2} + \frac{y}{x^3} = \frac{1}{y^2} + yx^{-3}$$

$$\frac{\partial z}{\partial x} = 0 + yx - 3x^{-4}$$

$$= -\frac{3y}{x^4} = -\frac{3\times1}{1} = -3$$

$$\frac{\partial z}{\partial y} = -2y + yx - 0 = -2+1 = -1$$

② If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Ans. → we have,

$$u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$u = x^2(y-z) - y^2(x-z) + z^2(x-y)$$

Diff. partially w.r.t. x, keeping y and z constants we get

$$\frac{\partial u}{\partial x} = (y-z) \times 2x - y^2(1-0) + z^2(1-0)$$

$$\frac{\partial u}{\partial x} = 2xy - 2xz - y^2 + z^2 \quad — ①$$

similarly $\frac{\partial u}{\partial y} = x^2(1-0) - (x-z)2y + z^2(0-1)$

$$\frac{\partial u}{\partial y} = x^2 - 2xy + 2yz - z^2 \quad — ②$$

similarly $\frac{\partial u}{\partial z} = x^2(-1) - y^2(0-1) + (x-y)2z$

$$\frac{\partial u}{\partial z} = -x^2 + y^2 + 2xz - 2yz \quad — ③$$

① + ② + ③

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{2xy} - \cancel{2xz} - \cancel{y^2} + \cancel{z^2} + \cancel{x^2} - \cancel{2xy} + \cancel{2yz} - \cancel{z^2}$$
$$- \cancel{x^2} + \cancel{y^2} + \cancel{2xz} - \cancel{2yz} = 0$$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ proved

(3) If $u = \frac{x^2+y^2}{x+y}$, prove that

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left\{1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right\}.$$

Ans. →

$$u = \frac{x^2+y^2}{x+y}$$

Diff. partially w.r.t. x keeping y constant

$$\frac{\partial u}{\partial x} = \frac{2x(x+y) - 1\cdot(x^2+y^2)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

similarly $\frac{\partial u}{\partial y} = \frac{2y(x+y) - 1\cdot(x^2+y^2)}{(x+y)^2} = \frac{2xy+y^2-x^2}{(x+y)^2}$

$$\therefore \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{2xy+y^2-x^2}{(x+y)^2}$$

$$= \frac{x^2+\cancel{2xy}-y^2-\cancel{2xy}+y^2-x^2}{(x+y)^2} = \frac{2(x^2-y^2)}{(x+y)^2} = \frac{2(x-y)}{(x+y)}$$

$$\therefore L.H.S. = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \left(\frac{2(x-y)}{(x+y)}\right)^2 = \frac{4(x-y)^2}{(x+y)^2}$$

Again $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{x^2+2xy-y^2}{(x+y)^2} + \frac{2xy+y^2-x^2}{(x+y)^2}$

$$= \frac{\cancel{x^2}+2xy-\cancel{y^2}+2xy+\cancel{y^2}-\cancel{x^2}}{(x+y)^2} = \frac{4xy}{(x+y)^2}$$

$$R.H.S. = 4\left\{1 - \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\right\} = 4\left\{1 - \frac{4xy}{(x+y)^2}\right\}$$

$$= 4\frac{(x+y)^2 - 4xy}{(x+y)^2} = 4\frac{(x-y)^2}{(x+y)^2}$$

$\therefore$ L.H.S. = R.H.S. proved.

④ QN: → If $u = a\sin\frac{x}{y} + b\cos\frac{y}{x}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Ans. → we have; $u = a\sin\frac{x}{y} + b\cos\frac{y}{x}$

$$\therefore \frac{\partial u}{\partial x} = a\cos\frac{x}{y} \times \frac{1}{y} + b\sin\frac{y}{x} \times \frac{y}{x^2}$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{x}{y}a\cos\frac{x}{y} + \frac{y}{x}b\sin\frac{y}{x} \quad -①$$

$$\frac{\partial u}{\partial y} = a\cos\frac{x}{y} \times \left(-\frac{x}{y^2}\right) - b\sin\frac{y}{x} \times \frac{1}{x}$$

$$\therefore y\frac{\partial u}{\partial y} = -\frac{x}{y}\cdot a\cos\frac{x}{y} - \frac{y}{x}b\sin\frac{y}{x} \quad -②$$

① + ②

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{y}a\cos\frac{x}{y}} + \cancel{\frac{y}{x}b\sin\frac{y}{x}} - \cancel{\frac{x}{y}a\cos\frac{x}{y}}$$
$$- \cancel{\frac{y}{x}b\sin\frac{y}{x}} = 0$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

⑤ Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$, when $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$.

Q, If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Ans. → we have; $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} \times y \times -\frac{1}{x^2}$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \times \frac{1}{y} - \frac{1}{\frac{x^2+y^2}{x^2}} \frac{y}{x^2}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \times \frac{1}{\cancel{y}} - \frac{\cancel{x^2}}{x^2+y^2} \times \frac{y}{\cancel{x^2}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{——(1)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times x \times -\frac{1}{y^2} + \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \times 1$$

$$\frac{\partial u}{\partial y} = -\frac{y}{\sqrt{y^2-x^2}} \times \frac{x}{\cancel{y^2}\,y} + \frac{\cancel{x^2}\,x}{x^2+y^2} \times \frac{1}{\cancel{x}}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad -(2)$$

(1) + (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ Proved.

(6) QN. → If $u = \log\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

Ans. → $\because u = \log\frac{x^2+y^2}{x+y} = \log(x^2+y^2) - \log(x+y)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \times 2x - \frac{1}{x+y} \times 1$$

$$x\frac{\partial u}{\partial x} = \frac{2x^2}{x^2+y^2} - \frac{x}{x+y} \quad -(1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \times 2y - \frac{1}{x+y} \times 1 = \frac{2y}{x^2+y^2} - \frac{1}{x+y}$$

$$y\frac{\partial u}{\partial y} = \frac{2y^2}{x^2+y^2} - \frac{y}{x+y} \quad -(2)$$

(1) + (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^2}{x^2+y^2} - \frac{x}{x+y} + \frac{2y^2}{x^2+y^2} - \frac{y}{x+y}$$

$$= \frac{2}{\cancel{x^2+y^2}}\left[\cancel{x^2+y^2}\right] - \frac{1}{\cancel{x+y}}\left[\cancel{x+y}\right] = 2-1 = 1.$$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$ Proved.